Parametrices for Degenerate Operators of Grushin's Type (PARTIAL DIFFERENTIAL EQUATIONS AND THEIR APPLICATIONS)

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Introduction. Grushin studied the hypoellipticity of an degenerate operator A of the form given in §1. In his paper [2] he used operator valued pseudo-differential operators essentially.

In this note we construct a left parametrix Q for the operator A as a pseudo-differential operator by symbol calculus instead of his method. For the construction of Q, we use the fundamental solution of parabolic equation studied in [4] and [5]. Also we discuss estimates for A.

We give main theorems and several examples in §1. In §2 and §3 the method of construction of Q will be shown. We devote §4 for prooves of Theorem B and Theorem C.

§1. Main theorems and examples.

In this note we treat operators defined in R^3 for simplicity. Consider an operator

$$(1)\quad A= A(x_1,y,D_{x_1},D_{x_2},D_y) = \sum_{\substack{|\alpha|\leq m \\ (\sigma,\gamma)\geqq(\tau,\alpha)-m}} a_{\alpha,\gamma} x_1^{\gamma_1} y^{\gamma_3} D_{x_1}^{\alpha_1} D_{x_2}^{\alpha_2} D_y^{\alpha_3}$$

where $a_{\alpha,\gamma}$ are constants, $D_j = \partial/\partial(ix_j)$ $\alpha=(\alpha_1,\alpha_2,\alpha_3)$, $\tau=(\tau_1,\tau_2,1)$,

$\tau_j \geqq 1$ $(j=1,2)$, $\gamma=(\gamma_1, o, \gamma_3)$, $\sigma=(\sigma_1, o, 1)$, $\min(\tau_1,\tau_2)>\sigma_1>0$

and $(\tau,\alpha) = \sum_{j=1}^{3} \tau_j \alpha_j$

Let A_o be the principal part of A in a sense, that is,

$$(2)\quad A = A(x_1,y,D_{x_1},D_{x_2},D_{x_3}) = \sum_{\substack{|\alpha|\leq m \\ (\sigma,\gamma)=(\tau,\alpha)-m}} a_{\alpha,\gamma} x_1^{\gamma_1} y^{\gamma_3} D_{x_1}^{\alpha_1} D_{x_2}^{\alpha_2} D_y^{\alpha_3}$$

We denote by $A(x_1,y,\xi,\eta)$ the differential polynomial correspond to A_o.

Condition 1. $A_o(x_1,y,D_{x_1},D_{x_2},D_y)$ is elliptic for $|x_1|+|y|\neq o$ i.e.

$$A_o(x_1,y,\xi,\eta)\neq 0$$

for $|x_1|+|y|\neq 0$, $\xi\in R^2$, $\eta\in R^1$, and $|\xi|+|\eta|\neq 0$.

Condition 2. For all $x_1\in R^1$ and for all nonzero vector $\xi\in R^2$ the equation $A_o(x_1,y,\xi,D_y)v(y)=0$ has no nonzero solution in $\mathcal{S}(R^1)$.

We get main theorems as follows.

Theorem A. Under condition 1 and condition 2, there exist a neighbourhood Ω of $x_1=y=0$ and a left parametrix Q for A in a class of pseudo-differential operators;

QA= I+W in Ω, where W is a smoothing operator.

By the construction of Q we get estimates for A and A_0.

$$(3)_s \qquad \|u\|_{m+s} \leqq C_K(\|A_0 u\|_s + \|u\|_s) \qquad \forall u \in C_0^\infty(K),$$

$$(4)_s \qquad \|u\|_{(m+s)} \leqq C(\|Au\|_{(s)} + \|u\|_{(s)}) \qquad \forall u \in C_0^\infty(\Omega),$$

where $\|\cdot\|_m$ is the usual Sobolev norm, K is any compact set of such that $K \cap \{0,x_2,0\} = \phi$, $\|u\|_{(m)} = \|\Lambda_{\tau}^m(D)u\|_0$ and Λ_τ is a pseudo-differential oparator with symbol $\sum_{j=1}^{2} |\xi_j|^{1/\tau_j} + |\eta| + 1$.

Moreover we get the following statement.

"If $Au \in H^s_{loc}(\Omega)$ and $u \in \mathcal{D}'(\Omega)$, then $u \in H^{s+m/\bar{\tau}}_{loc}(\Omega)$

where $\bar{\tau} = \max(\tau_1, \tau_2)$.

Remark. If $\bar{\tau}=1$, then condition 1 means that A is elliptic in case $|x_1| + |y|$ is sufficient small.

We may assume $\bar{\tau} > 1$ by the above remark.

Theorem B. If $(4)_0$ hold, then we obtain the following estimate for all $x_1 \in R^1$ and nonzero $\xi \in R^2$.

$$\|v\| \leqq C \|A_0(x_1, y, \xi, D_y)v\| \qquad \forall\, v \in \mathcal{S}(R^1_y)$$

where $\|\cdot\|$ is the norm in $L_2(R^1_y)$.

Owing to the above theorem we get

Theorem C. Assume that A satisfies $(3)_0$ and $(4)_0$. Then, A holds condition 1 and condition 2 .

Now let us give some examples

Example 1. $A=A_0=(-\Delta_y)^{\ell} + y^{2k}(-\Delta_{x_2})^{\ell}$ in R^2, where ℓ and k are positive integers. We can take $\tau=(1+k/\ell, 1)$, $\sigma=(0, 1)$ and m=2 . In this case condition 1 and condition 2 hold.

Example 2. (Grushin [2]). $A=A_0=\frac{\partial^2}{\partial y^2} + (y^2+x_1^2)\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}\right) + i\lambda\frac{\partial}{\partial x_2}$. In this case we can choose $\tau=(2,2,1)$, $\sigma=(1,0,1)$ and m=2. Condition 1 holds. Condition 2 holds if $|\lambda|<1$ or $\mathrm{Im}\lambda \neq 0$.

Example 3. $A=A_0=\left(\frac{\partial}{\partial y}-iay^k\frac{\partial}{\partial x_2}\right)\left(\frac{\partial}{\partial y}-iby^k\frac{\partial}{\partial x_2}\right) + icy^{k-1}\frac{\partial}{\partial x_2}$ in R^2, where k is a odd integer, a,b and c are real constants such that $ab<0$. In this case $\tau=(k+1, 1)$, $\sigma=(0, 1)$ and m=2. Condition 1 always holds. Condition 2 holds if and only if

$$\frac{c}{a-b} \not\equiv 0,1 \pmod{(k+1)}.$$

This result is due to Gilioli and Treves [1].

§ 2. Notations and theorems for pseudo-differential operators

For a pair of real vectors $\rho=(\rho_1,\rho_2,\cdots,\rho_n)$ and $\delta=(\delta_1,\delta_2,\cdots,\delta_n)$ we say $\rho>\delta$ if $\rho_j>\delta_j$ for all j. In this section we fix ρ and δ such that $\rho>\delta\geqq 0$.

Definition (cf. [5]). We say that a C^∞-function $\lambda(x,\xi)$ define in $R^n_x\times R^n_\xi$ is a basic weight function when $\lambda(x,\xi)$ satisfies the conditions below:

$$(5)\qquad \left|\lambda^{(\alpha)}_{(\beta)}(x,\xi)\right| < A_{\alpha,\beta}\,\lambda(x,\xi)^{1-(\rho,\alpha)+(\delta,\beta)}$$

$$(6)\qquad 1\leqq\lambda(x+y,\xi)\leqq A\langle y\rangle^{\tau_0}\lambda(x,\xi) \qquad (\tau_0\geqq 0),$$

where $\lambda^{(\alpha)}_{(\beta)}(x,\xi)=(\partial/\partial\xi_1)^{\alpha_1}\cdots(\partial/\xi_n)^{\alpha_n}(-i\partial/\partial x_1)^{\beta_1}\cdots(-i\partial/\partial x_n)^{\beta_n}$ $\lambda(x,\xi)$ and $\langle y\rangle=(|y|^2+1)^{1/2}$.

We denote by $S^m_{\lambda,\rho,\delta}(-\infty<m<\infty)$ the set of all C^∞-functions $p(x,\xi)$ defined in $R^n_x\times R^n_\xi$ which satisfies for any α and β

$$\left|p^{(\alpha)}_{(\beta)}(x,\xi)\right|\leqq C_{\alpha,\beta}\,\lambda(x,\xi)^{m-(\rho,\alpha)+(\delta,\beta)}$$

for some constants $C_{\alpha,\beta}$. For a symbol $p(x,\xi)\in S^m_{\lambda,\rho,\delta}$ we define a pseudo-differential operator by

$$Pu(x)=p(x,D_x)u(x)=\int e^{ix\cdot\xi}p(x,\xi)\hat{u}(\xi)\,đ\xi\ ,$$

where $đ\xi=(2\pi)^{-n}d\xi$ and $\hat{u}(\xi)$ denote the Fourier transform of $u(x)$ in $\mathcal{S}$ defined by

$$\hat{u}(\xi)=\int e^{-ix\cdot\xi}u(x)dx.$$

We call a pseudo-differential operator a smoothing operator when its symbol belongs to $S^m_{\langle\xi\rangle,\rho,\delta}$ for any m.

Now consider a pseudo-differential operator of parabolic type.

$$L=\frac{\partial}{\partial t}+p(x,D_x)$$

We call an operator E(t) a fundamental solution for L when E(t) satisfies

$$\begin{cases} L\,E(t)=0 & \text{in}\quad 0<t<\infty\ , \\ E(0)=I\ . \end{cases}$$

By [5], we get the next theorem

Theorem 1. Assume that $p(x,\xi)\in S^{\ell}_{\lambda,}$, satisfies

$$\operatorname{Re} p(x,\xi)+c\geqq c_o\lambda(x,\xi)^{\ell}$$

for positive constants c and c_o. Then, there exists a fundamental solution E(t) which belongs to $S^{0}_{\lambda,\rho,\delta}$ with parameter t and whose symbol $e(t;x,\xi)$ has the following expansion

$$e(t;x,\xi) = \sum_{j=0}^{N-1} e_j(t;x,\xi) + r_N(t;x,\xi)$$

where $e_j(t;x,\xi) \in S^{-\omega j}_{\lambda,\rho,\delta}$, $r_N(t;x,\xi) \in S^{\ell-\omega N}_{\lambda,\rho,\delta}$, $\omega = \min(\rho_j - \delta_j)$ and N is any number such that $\omega N \geqq \ell$. Moreover we get $e_0(t;x,\xi) = \exp\left\{-tp(x,\xi)\right\}$ and

$$e_{j(\beta)}^{(\alpha)}(t;x,\xi) = a_{j,\alpha,\beta}(t;x,\xi)\, e_0(t;x,\xi) \qquad (j \geqq 1),$$

where $a_{j,\alpha,\beta}(t;x,\xi)$ satisfy

$$\left|a_{j,\alpha,\beta}(t;x,\xi)\right| \leqq C_{j,\alpha,\beta}\, \lambda(x,\xi)^{-\omega j-(\rho,\alpha)+(\delta,\beta)}$$

$$\times \sum_{k=2}^{|\alpha|+|\beta|+2j} \left\{t \operatorname{Re} p(x,\xi)\right\}^k$$

About the behavior of $e(t;x,\xi)$ for large t, we get

Theorem 2 (Tsutsumi [5]). Let $p(x,\xi)$ satisfy the assumption of theorem 1 and

$$\operatorname{Re}(p(x,D_x)u,\ u) \geqq c_1 \|u\|^2 \qquad u \in \mathcal{S}(R^n)$$

with a positive constant c_1. Moreover let $\rho > 0$ and $\lambda(x,\xi)$ satisfy

$$\lambda(x,\xi) \geqq a(|\xi| + |x| + 1)^a$$

for some positive constant a. Then the symbol $e(t;x,\xi)$ constructed in theorem 1 holds the following estimate for any integers j and k and positive constant ε

$$\lambda(x,\xi)^j\left|\frac{\partial^k}{\partial t^k}e^{(\alpha)}_{(\beta)}(t+\varepsilon;x,\xi)\right|\leqq C(j,k,\alpha,\beta,\varepsilon)\exp(-c_2 t) \quad (t\geqq 0)$$

for any α and β ,

where c_2 is any constant less than c_1 and $C(j,k,\alpha,\beta,\varepsilon)$ is independent of t .

Now assume that the basic weight function $\lambda(x,\xi)$ is independent of x and $b^{-1}|\xi|^{k_1} \leqq \lambda(\xi) \leqq b\,|\xi|^{k_2}$ $\quad (k_1, k_2 > 0)$. We denote the Sobolev norm $\|\cdot\|_{m,\lambda}$ by

$$\|u\|_{m,\lambda} = \|\lambda(D_x)^m u\|$$

where $\|\cdot\|$ is the norm in $L^2(R^n)$.

. Let Ω be a open set in R^n and let $p(x,\xi)$ hold the estimate below

$$(7) \qquad \left|p^{(\alpha)}_{(\beta)}(x,\xi)\right| \leqq C_{K,\alpha,\beta}\,\lambda(\xi)^{M-(\rho,\alpha)+(\delta,\beta)} \qquad \forall x\in K$$

for any compact set K in Ω .

Theorem 3 (Tsutsumi [6]). Let $P=p(x,D_x)$ satisfy (7) and

$$\|u\|^2_{m,\lambda} \leqq C_K(\|Pu\| + \|u\|_{m',\lambda}) \qquad u\in C^\infty_0(K).$$

where $M\geqq m\geqq 0$, $m>m'$ and K is any compact set in Ω . Then, for every K and every integer N one can find C so that when $x\in K$ and $\xi\in R^n$

$$\lambda(\xi)^{2m}\|\psi\|^2 \leqq C\int\Big|\sum_{|\alpha+\beta|<N} p^{(\alpha)}_{(\beta)}(x,\xi)\,\lambda(\xi)^{(\theta,\alpha-\beta)} y^{\beta} D_y^{\alpha}\psi/\alpha!\,\beta!\Big|^2 dy$$

$$+\lambda(\xi)^{2M-2\varepsilon_0 N+d_1}\Big(\sum_{|\alpha+\beta|\leq N+d_2}\int |y^{\beta} D_y^{\alpha}\psi|^2 dy\Big) \quad , \psi\in C_0^{\infty}(R^n)$$

Here $\theta=(\theta_1,\theta_2,\cdots,\theta_n)$ is chosen such that $\varepsilon_0=\min(\rho_j-\theta_j,\ \theta_j-\delta_j,\ \theta_j)>0$ and both d_1 and d_2 are constants independant of N .

§ 3. Outline of construction of a left parametrix Q.

We introduce some notations.

$$h=h(x_1,y)=(|x|^{1/\sigma_1}+|y|)$$

$$\nu=\nu(x_1,y,\xi,\eta)=\sum_{j=1}^{2} h(x_1,y)^{\tau_j-1}|\xi_j|+|\eta|$$

$$\mu=\mu(\xi)=\sum_{j=1}^{2}|\xi_j|^{1/\tau_j}+1$$

$$\chi=\chi(\xi,\eta)=\sum_{j=1}^{2}|\xi_j|+|\eta|+1$$

We denote by $S^m(\nu,\mu,\chi)$ $(-\infty<m<\infty)$ the set of all C^{∞}-functions $p(x_1,y,\xi,\eta)$ defined in $R^1_{x_1}\times R^1_y\times R^2_{\xi}\times R^1_{\eta}$ which satisfy

(8) $$\Big|p^{(\alpha)}_{(\beta)}(x_1,y,\xi,\eta)\Big|\leqq C_{\alpha,\beta}(\nu+\mu)^{m-|\beta_3|}\mu^{|\beta_3|+\sigma_1|\beta_1|}\chi^{-|\alpha_1|-|\alpha_2|}$$

for any $\alpha=(\alpha_1,\alpha_2,\alpha_3)$, $\beta=(\beta_1,0,\beta_3)$. We define a psuedo-differntial operator $Pu(x_1,x_2,y)$ with a symbol $\sigma(P)=p(x_1,y,\xi,\eta)\in S^m(\nu,\mu,\chi)$ by

$$Pu(x_1,x_2,y)=\int e^{i(x_1\xi_1+x_2\xi_2+y\eta)}p(x_1,y,\xi,\eta)\hat{u}(\xi,\eta)d\xi d\eta$$

for $u\in\mathcal{S}(R^3)$.

We get the following lemma for operators of this class.

Lemma 1. (Tsutsumi [6]) (i) If P_j belongs to $S^{m_j}(\nu,\mu,\chi)$ $(j=1,2)$ then P_1P_2 belongs to $S^{m_1+m_2}(\nu,\mu,\chi)$ and

$$\sigma(P_1P_2)(x_1,y,\xi,\eta) - \sigma(p_1(x_1,y,\xi,D_y)p_2(x_1,y,\xi,D_y))$$

belongs to $S^{-\delta_0}(\nu,\mu,\chi)$, where $\delta_0=\min(\tau_1,\tau_2)-\sigma_1$.

(ii) If P belongs to $S^0(\nu,\mu,\chi)$, then P is a bounded operator in $L^2(R^3)$

(iii) $P\in S^m(\nu,\mu,\chi)$ has pseudo-local propety, i.e. For any $\varphi,\psi\in C^\infty(R^3)$ P is a smoothing operator when $\mathrm{supp}\varphi\cap\mathrm{supp}\psi=\phi$.

The main result of this section is the proposition below.

Proposition 1. Under condition 1 and condition 2 there exist a neighbourhood $\Omega'\subset R^2$ of $x_1=y=0$ and $\tilde{A}(x_1,y,\xi,\eta)\in S^m(\nu,\mu,\chi)$ which satisfy the following properties.

(i) We can construct a left parametrix $Q\in S^{-m}(\nu,\mu,\chi)$ for $\tilde{A}$, i.e.

$$Q\tilde{A} = I + W, \quad \text{where } w \text{ is a smoothing operator .}$$

(ii) $(A-\tilde{A})(x_1,y,\xi,\eta)= 0 \qquad \text{if} \quad (x_1,y)\in\Omega'$.

By this proposition and lemma 1 one can prove theorem A, noting that Λ_τ belongs to $S^1(\nu,\mu,\chi)$ and that Q is elliptic for $|x_1|+|y|\neq 0$, i.e. The symbol $q(x_1,y,\xi,\eta)$ of Q satisfy $|q|\geq c(|\xi|+|\eta|+1)^{-m}$ for $|x_1|+|y|\neq 0$.

We need several steps to show proposition 1.

Lemma 2. For all α and β we have

$$\left|\partial_\eta^\alpha \partial_y^\beta A_o(x_1, y, \xi, \eta)\right| \leqq C_{\alpha,\beta} (\nu+\mu)^{m-|\alpha|} \mu^{|\beta|} \qquad \text{for } |\xi| \geqq c > 0$$

Moreover if we fix a $C_0^\infty(R^1)$-function φ, then $\varphi(|x_1|^2+|y|^2) A_o(x_1, y, \xi, \eta)$ belongs to $S^m(\nu, \mu, \chi)$ and $\varphi(|x_1|^2+|y|^2)(A-A_o)(x_1, y, \xi, \eta)$ satisfies (8) for $C_{\alpha,\beta}\, h(x_1, y)^{\varepsilon_1}$ $(\varepsilon_1 > 0)$ instead of the constants $C_{\alpha,\beta}$ in (8).

Lemma 3. If condition 1 hold , then we get for some constant $C > 0$

$$C^{-1} \left|A_o(x_1, y, \xi, \eta)\right| \leqq \nu(x_1, y, \xi, \eta) \leqq C \left|A_o(x_1, y, \xi, \eta)\right| .$$

Lemma 4. (Grushin [2]). We find a constant $C_1 > 0$ such that

$$\|u\| \leqq C_1 \|A_o(x_1, y, \xi, D_y) u\| \qquad \forall u \in \mathcal{S}(R^1)$$

for any ξ when $|\xi|=1$.

Set $\lambda_z(y, \eta)$ with parameters $z=(x_1, \xi)$ by

$$\lambda_z(y, \eta) = \left(|A_o(x_1, y, \xi, \eta)|^2 + 1\right)^{1/2m}$$

then by lemma 2 and lemma 3, we get

Lemma 5. If $|\xi|=1$, then we get

(i) $\lambda_z(y, \eta)$ is a basic weight function which holds (5) and (6) for $A_{\alpha,\beta}$ and A independant of parameters z.

(ii) $|A_o(x_1, y, \xi, \eta)|^2 + 1 \geqq \lambda_z(y, \eta)^{2m}$

(iii) $\left|\partial_\eta^\alpha \partial_y^\beta A_o(x_1, y, \xi, \eta)\right| \leqq C_{\alpha,\beta}\, \lambda_z(y, \xi)^{m-|\alpha|}$

where $C_{\alpha,\beta}$ are independant of z .

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Now, by lemma 5 we can apply theorem 1 for $L_z = L_{x_1,\xi}$ $(|\xi|=1)$ defined by

$$L_z = L_{x_1,\xi} = \frac{\partial}{\partial t} + p(x_1,y,\xi,D_y) ,$$

where $p(x_1,y,\xi,D_y)=A_o(x_1,y,\xi,D_y)\, A_o^*(x_1,y,\xi,D_y)$ and $A_o^*(x_1,y,\xi,D_y)$ is the adjoint operator of $A(x_1,y,\xi,D_y)$.

Let $e(t;y,D_y;z) \in S^o_{\lambda_z,1,0}$ be the fundamental solution for L_z with parameters z. $(|\xi|= 1)$.

For any ξ $(|\xi|\geqq 1)$ we get

Lemma 6. If condition 1 and condition 2 hold, then there exists the fundamental solution $u(t;y,D_y;z)$ for L_z which admits the expansion below for $N\geqq 2m$,

$$u(t;y,\eta;z) = \sum_{j=0}^{N-1} u_j(t;y,\eta;z) + v_N(t;y,\eta;z)$$

$$|\partial_\eta^\alpha \partial_y^\beta u_j(t;y,\eta;z)| \leqq C_{j,\alpha,\beta}(\nu+\mu)^{-j-|\alpha|}\, \mu^{j+|\beta|} \exp[-c_o(\nu+\mu)^{2m}t]$$

$$|\partial_\eta^\alpha \partial_y^\beta v_N(t;y,\eta;z)| \leqq C_{N,\alpha,\beta}(\nu+\mu)^{-N-|\alpha|}\, \mu^{N+|\beta|} ,$$

(9) $$|\partial_\eta^\alpha \partial_y^\beta u(t;y,\eta;z)| \leqq C'_{\alpha,\beta}(\nu+\mu)^{-k-|\alpha|} \mu^{k+|\beta|} \exp[-c_o \mu^{2m}t]$$

for $\mu^m t \geqq \varepsilon > 0$ and all k.

Proof. For $|\xi|=1$ by lemma 4 and theorem 2 , one can find a constant C independent of z such that

$$\max_{|\alpha|+|\alpha'|+|\beta|+|\beta'|<\ell,\ y\in R^1,\ \eta\in R^1} \sup |y^\alpha \eta^{\alpha'} D_y^\beta D_\eta^{\beta'} e(t+\varepsilon;y,\eta;z)| \leqq C\exp[-C_2 t] \quad (t\geqq 0)$$

for any $\varepsilon>0$, ℓ and $C_2<C_1$.

$A_o(x_1,y,\xi,\eta)$ is quasihomogeneous in the sense

$$A_0(\lambda^{-\sigma_1}x_1, \lambda^{-1}y, \lambda^{\tau_1}\xi_1, \lambda^{\tau_2}\xi_2, \lambda\eta) = \lambda^m A_0(x_1, y, \xi_1, \xi_2, \eta)$$

then $u(t;y,\eta;z)$ is given by

(10) $$u(t;y,\eta;x_1,\xi_1,\xi_2)=e(\tilde{\mu}^{2m}t;\tilde{\mu}y,\tilde{\mu}^{-1}\eta;\tilde{\mu}^{\sigma_1}x_1, \tilde{\mu}^{-\tau_1}\xi_1,\tilde{\mu}^{-\tau_2}\xi_2)$$

where $\tilde{\mu}=\tilde{\mu}(\xi)$ is the positive root of the equation $\sum_{j=1}^{2}\tilde{\mu}^{-2\tau_j}\xi_j^2=1$.
By (10) and theorem 1 we get the assertion.

Put $r(x_1,y,\xi,\eta)=r(y,\eta;x_1,\xi)=\int_0^\infty u(t;y,\eta;x_1,\xi)dt$ for $|\xi|\geqq 1$.
Then $R(x_1,\xi)=r(y,D_y;x_1,\xi)$ is a left and right inverse of $p(x_1,y,\xi,D_y)$. Let us define a pseudo-differential operator $K(x_1,\xi)$ with parameters by

$$K(x_1,\xi)=R(x_1,\xi)A_0(x_1,y,\xi,D_y) .$$

Then for $|\xi|\geqq 1$,

Lemma 7. $K(x_1,\xi)$ is a left inverse of $A_0(x_1,y,\xi,D_y)$ with a symbol $k(y,\eta;x_1,\xi)=k(x_1,y,\xi,\eta)$ which admits for any α and β

$$\left|\partial_\eta^\alpha \partial_y^\beta \; k(y,\eta;x_1,\xi)\right| \leqq C_{\alpha,\beta}(\nu+\mu)^{-m-|\alpha|}\mu^{|\beta|}.$$

Moreover $\varphi(|x_1|^2+|y|^2)k(x_1,y,\xi,\eta)$ belongs to $S^{-m}(\nu,\mu,\chi)$ for any $\varphi\in C_0^\infty(R^1)$.

Proof. It is sufficient to show that $r(y,\eta;x_1,\xi)$ satisfies

$$\left|\partial_\eta^\alpha \partial_y^\beta \; r(y,\eta;x_1,\xi)\right|\leqq C'_{\alpha,\beta}(\nu+\mu)^{-2m-|\alpha|}\mu^{|\beta|}$$

and that $\varphi(|x_1|^2+|y|^2)r(x_1,y,\xi,\eta)$ belongs to $S^{-2m}(\nu,\mu,\chi)$, by lemma 2.

We write

$$r(y,\eta;x_1,\xi)=\int_0^{\mu^{-m}(\xi)} u(t;y,\eta;x_1,\xi)dt + \int_{\mu^{-m}(\xi)}^{\infty} u(t;y,\eta;x_1,\xi)dt$$
$$= I_1 + I_2 .$$

For I_1 we fix N such that $N \geqq 2m$ and use the expansion in lemma 6. We apply (9) taking k=2m for I_2 . For derivatives with respect to parameters x_1 and ξ, we can write for example

$$r_{\xi_j}(y,D_y;x_1,\xi)=-r(y,D_y;x_1,\xi)p_{\xi_j}(x_1,y,\xi,D_y)r(y,D_y;x_1,\xi)$$

Then, by lemma 2 and symbol calculus of pseudo-differential operator with parameters $z=(x_1,\xi)$ and also noting that $R(x_1,\xi)$ has pseudo-local property with respect to y, one get the assertion.

By lemma 2 we can construct $\widetilde{A_o}(x_1,y,\xi,\eta)\in S^m(\nu,\mu,\chi)$ such that $|\widetilde{A_o}(x_1,y,\xi,\eta)|\geqq c\chi^m$ for $|x_1|+|y|\geqq 1$ and $\widetilde{A_o}(x_1,y,\xi,\eta)=A_o(x_1,y,\xi,\eta)$ if $|x_1|+|y|\leqq 1$. Now let us construct a pseudo-differential operator $Q_o=q_o(x_1,y,D_{x_1},D_{x_2},D_y)$ which is a left parametrix for $\widetilde{A_o}(x_1,y,D_{x_1},D_{x_2},D_y)$ by using $k(x_1,y,\xi,\eta)$.

Put $q_o'(x_1,y,\xi,\eta)=\varphi_1(|x_1|^2+|y|^2)\varphi_2(|\xi|^2)k(x_1,y,\xi,\eta)+\varphi_1(|x_1|^2+|y|^2)\varphi_1(|\xi|^2)$
$\varphi_2(|\xi|^2+|\eta|^2)k_1(x_1,y,\xi,\eta)+\varphi_2(|x_1|^2+|y|^2)k_1(x_1,y,\xi,\eta)$.

Here $k_1(x_1,y,\xi,\eta)=A_o(x_1,y,\xi,\eta)^{-1}$, $\varphi_1\in C_o^\infty(R^1)$ such that $\varphi_1(t)=1$ $|t|\leqq 1$

$\varphi_1(t)=0$ $|t|\geq 2$ and $\varphi_2=1-\varphi_1$. It is clear that $q_o'(x_1,y,\xi,\eta)$ belongs $S^{-m}(\nu,\mu,\chi)$.

$$q_o'(x_1,y,\xi,D_y)\widetilde{A_o}(x_1,y,\xi,D_y)-I \in S^{-\delta_o'}(\nu,\mu,\chi) \tag{11}$$

where $\delta_o'=\min(\delta_0,1)>0$ using (i) of lemma 1 .

From (11) one can get a left parametrix $Q_o\in S^{-m}(\nu,\mu,\chi)$.

For any $\varepsilon>0$ there exist Ω' and $\widetilde{A}(x_1,y,\xi,\eta)\in S^m(\nu,\mu,\chi)$ such that

$$\widetilde{A}(x_1,y,\xi,\eta) = A(x_1,y,\xi,\eta) \qquad \text{if } (x_1,y)\in\Omega'$$

$$\left|(\widetilde{A}-\widetilde{A_o})^{(\alpha)}_{(\beta)}(x_1,y,\xi,\eta)\right| \leqq \varepsilon\ (\nu+\mu)^{m-|\alpha_3|}\ \mu^{\sigma_1|\beta_1|+|\beta_3|}\ \chi^{-|\alpha_1|-|\alpha_2|} \tag{12}$$

for $\alpha=(\alpha_1,\alpha_2,\alpha_3)$ and $\beta=(\beta_1,0,\beta_3)$.

If ε is sufficient small, then by (12) there exists $R_o\in S^o(\nu,\mu,\chi)$ the inverse of $I+Q_o(\widetilde{A}-\widetilde{A_o})$. Set $Q=R_oQ_o$, then Q is a required left parametrix .

§ 4.Prooves of theorem B and theorem C.

At first we show theorem B.

Define a weight function $\lambda(\xi,\eta)$ by

$$\lambda(\xi,\eta) = \varphi(\xi,\eta)\Big(\sum_{j=1}^{2}|\xi_j|^{1/\tau_j}+\ \ |\eta|^{1/\bar{\tau}}\Big)\ +1$$

where $\varphi\in C^\infty(R^3)$ such that $\varphi(\zeta)=1$ ($|\zeta|\geq 1$) , $\varphi(\zeta)=0$ ($|\zeta|\leq 1/2$) and $\bar{\tau}=\max(\tau_1,\tau_2)$. Then by $(4)_o$ it is clear that

$$\|u\|_{m,\lambda}\leqq C(\|Au\|+\|u\|) \qquad \forall u\in C_o^\infty(\Omega). \tag{13}$$

2

It is easy to see that $\widetilde{A}(x_1,y,\xi,\eta)$ belongs to $S^m_{\lambda,\rho,\delta}$ if $\rho=(\tau_1,\tau_2,\overline{\tau})$ and $\delta=(0,0,0)$. Now from (13) we can apply theorem 3 in §1 for $l=0$, choosing θ_j such that $\theta_j=(\tau_j+\sigma_j)/2$ $(j=1,2)$ and $\theta_3=1$.

Then for any N

$$\lambda(\xi,0)^{2m}\|\psi\|^2 \leqq C\Big\{\int\Big|\sum_{|\alpha+\beta|<N} A^{(\alpha)}_{(\beta)}(x_1,y,\xi,0)\,\lambda(\xi,0)^{(|\alpha_1|-|\beta_1|)\theta_1+(|\alpha_2|-|\beta_2|)\theta_2+|\alpha_3|-|\beta_3|} \tag{14}$$

$$\zeta^\beta D^\alpha_\zeta\psi(\zeta)/\alpha!\,\beta!\Big|^2 d\zeta+\lambda(\xi,0)^{2m\overline{\tau}-2\varepsilon_0 N+d_1}\sum_{|\alpha+\beta|\leqq N+d_2}\|\zeta^\beta D^\alpha_\zeta\psi\|^2\Big\}$$

Now take N such that $2m\overline{\tau}-2\varepsilon_0 N+d_1=2m+\varepsilon_0$. Note that (14) is true for $\mu(\xi)$ instead of $\lambda(\xi,0)$ and put $x_1=t^{-\sigma_1}x_1^0$, $y=0$ and $\xi_j=t^{\tau_j}\xi_j^0$ $(j=1,2)$ such that $\mu(\xi^0)=1$. Then quasihomogeneity of $\mu(\xi)$ and $A^{(\alpha)}_{(\beta)}(x_1,y,\xi,0)$ we get the following estimate

$$\|\psi\|^2 \leqq C\Big\{\int\Big|\sum_{|\alpha+\beta|<N} A_0{}^{(0,0,\alpha)}_{(0,0,\beta)}(x_1^0,0,\xi^0,0)\,\zeta^\beta D^\alpha_\zeta\psi/\alpha!\,\beta!\Big|^2 d\zeta\Big\}$$

$$+O(t^{-r})$$

where $r=\min(\varepsilon_1,(\tau_j-\sigma_j)/2)$.

Note that $\sum_{|\alpha+\beta|<N} A_0{}^{(0,0,\alpha)}_{(0,0,\beta)}(x_1^0,0,\xi^0,0)\,\zeta^\beta D^\alpha_\zeta\psi/\alpha!\,\beta!=A_0(x_1^0,\zeta,\xi^0,D_\zeta)\psi$ and quasihomogeneity of $A_0(x_1,y,\xi,\eta)$ we get the assertion of theorem B.

To show theorem C we take $\lambda'(\xi,\eta)=(\sum_{j=1}^{2}|\xi_j|^2+|\eta|^2+1)^{1/2}$.

Then by $(3)_0$ and theorem 3 we get for $(x_1,y)\in K$

$$\lambda(\xi,\eta)^m \leqq c_K |A_o(x_1,y,\xi,\eta)| \qquad \text{for } |\xi|+|\eta| \geqq c>0$$

By quasihomogeneity of $A_o(x_1,y,\xi,\eta)$, it is elliptic for $|x_1|+|y| \neq 0$.

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